

A TIME-HONOURED MYSTERY, FROM THE MENO OF PLATO

1. The dialogues of him whom most of us only know by his nickname Plato, the Broad, — broad of brow, some suppose, broad-minded, say others, and broad-shouldered, explain a third party, remembering his splendid physique and his youthful prowess in the arena, — dialogues, taken down, I suppose, on waxen tablets and subsequently engrossed on papyrus or sheepskin, long, long ago, copied by devotees, letter by letter, and assiduously commented for a score of centuries, have been, during the last five hundred years, printed in hundreds & thousands of copies, translated and explained by hundreds and thousands of professors and pupils, and still there remain passages which have baffled comprehension. In particular this is true of a celebrated paragraph in the



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Meno⁽¹⁾. Yet the passing manner in which the mathematical problem that has puzzled the commentators is referred to in the text makes one feel that it was one current at the time, ephemeral perhaps as a cross-word puzzle, and now fallen into complete, or partial, oblivion.

It was such considerations which led me to take up anew this much vexed question, and, I think, to solve it. I venture therefore to present to the Institute of Coimbra my version of the matter, the more so, because the study of this passage appears to illustrate the mind of the Master, and to throw light on the comparative state of mathematics at that time and at the present.

I shall not give any account of the earlier interpretations, I have conscientiously examined all that I could come across, and am only so far indebted to their learned inventors that I have gained considerable confidence in my Greek translation, which is quite literal. This is to be noted, since, though the commentators agree that there is no ground in the manuscripts for considering the text as in any way corrupt, they one and all alter the words to fit in with some preconceived interpretation of some hidden meaning. See foot note at the end.

2. The passage in question occurs where Socrates starts seriously on the discussion of Meno's question;

«Hast thou somewhat to tell me, oh Socrates, about whether virtue is to be taught, or not taught, but acquired by practice, or neither by practice nor by teaching, but is it just by nature that it comes to men, or how in the world».

The question was one, as Plato perceived, which could not be answered as it stood. It could be regarded from different points of view, and receive accordingly different so-

(1) Meno, 86 D-87 C

lutions. And, in point of fact, we find him in works which have come down to us, returning to the subject, and treating it in quite a different manner.

In the present case this is what Socrates, wisest of men, is represented as saying:

«If it was I that ruled, oh Meno, not only over myself, but also over thee, we should not have examined first of all whether virtue is, or is not, to be taught, before we had previously discovered what it is itself. But as thou dost not undertake to rule over thyself—because thou wouldest of course be a free man—but over me thou undertakest to rule, and thou dost rule over me, I will give way to thee. Well! what is to be done? we have, it seems, to examine whether something has certain properties, and we do not even know what it is! Anyway thou mightest relax a little the severity of thy rule over me, and agree to work *on a hypothesis* in discovering whether it can be taught or in what way. Now I use the term *on a hypothesis* in this sense, just as the geometricians often argue, when someone asks them, for instance, about an area, if it is possible to inset in this circle this triangular area, one of them would answer: I do not know off-hand whether this be so, but I think the assumption of some hypothesis which would be workable for our purpose would be the following. If, for instance, this area is such as, when spread out alongside the line given to it, to leave over just such an area as, if possible, that which was spread out may be, one thing appears to me to occur, and another if it is impossible to effect this. Having hypothecated this then, I will tell thee what happens about its inseting in the circle, if it is impossible or not.

«Just in the same way about our question of virtue, because we do not know either what it is or what sort of a thing it is, let us make a hypothesis about it and then see whether it is to be taught or not; saying, for instance, if virtue is

of such a sort as are the things which belong to the soul, will it, or will it not, be to be taught. First of all, however, if it is another sort of thing from knowledge, or the same thing, will it be to be taught, or not... Or is it not evident to all men that nothing else can a man be taught except knowledge?»?

Meno «That seems to me to be so».

Socrates. «If then virtue is knowledge, it is evident that it can be taught».

3. The type of mathematical problem to which Plato is here referring, was one which had occupied much the mathematicians of the school of Pythagoras, and the terms Plato uses seem to be borrowed from their vocabulary. It is not surprising that they were subsequently modified, and that in the works of Euclid, to whom we are indebted for most of our knowledge of early Greek mathematics, the expression

παρά τὴν γραμμὴν παρατείνειν,

which I render

to spread out alongside the line

— as Plato, with his poetic mind, might think of a sail spread out to catch the wind, being fixed by rings at the corners to two parallel ropes — becomes in Euclid a purely technical phrase

παρά τὴν γραμμὴν παραβάλλειν

usually translated

to apply to a line.

This expression proved one of the foremost stumbling-blocks to the commentators until quite lately, in spite of

the fact that Plato uses the same verb $\piαρατείνειν$ in the same sense in the Republic⁽¹⁾.

It was characteristic, however, that he did not take as an illustration the exact problem which the Pythagoreans had solved, but an analogous one.

The key to the comprehension of the passage is given by Proclus (410-485 A. D.) in his Commentary on Euclid. He says: «These things, says Eudemus, are ancient and are discoveries of the Muse of the Pythagoreans, I mean the application of areas ($\piαραβολή τῶν χωρίων$) their exceeding ($\υπερβολή$) and their falling short ($\ελλειψις$). It was from the Pythagoreans that later geometers took the names, which they again transferred to the so-called *conic* lines, designating one of these a parabola, (application), another a hyperbola (exceeding) and another an ellipse (falling short), whereas those godlike men of old saw the things signified by these names in the construction, in a plane, of areas upon a finite straight line. For, when you have a straight line set out and lay the given area exactly alongside the whole of the straight line, then they say that you *apply* ($\piαραβάλλειν$) the said area; when however you make the length of the area greater than the straight line itself, it is said to *exceed* ($\υπερβάλλειν$) and when you make it less, in which case, after the area has been drawn, there is some part of the straight line extending beyond it, it is said to *fall short* ($\ελλείπειν$). Euclid too, in his sixth book, speaks in this way both of *exceeding* and *falling short*». The Eudemus here quoted is the Eudemus of Rhodes, a pupil, as Simplicius (500 A. D.) tells us, of

(1) Republic VII, 527. In Adam's edition (Cambridge Univ. Press, 1907) Vol. 2, p. 118, note 7, the view here advocated is accepted. The first apparently to perceive this was Blass in his Bonn Dissertation of 1861, *De Platone Mathematico*, and his version, which is otherwise somewhat childish, is accepted by M. Cantor, but ruefully, in his *Geschichte der Mathematik* I, p. 188 of the 3d edition, also 1907.

Aristotle, who was himself a pupil of Plato. The passage before us of the *Meno* is therefore the earliest extant reference to an important mathematical theory, which, in early times, replaced the theory of quadratic equations in Algebra. Indeed, by means of propositions which are chronicled in Euclid, the Greeks were able to deal with any problem which we should nowadays express by a quadratic equation.

The problem to which Plato is referring in the present instance is not one of these, and it is interesting to find that it reduces to a biquadratic equation with at most two real roots and falls into line with other mathematical problems which we know to have occupied his mind.

4. There are several references in Plutarch (who died at a great age in 120 A. D.) to the proposition known as Euclid I, 44, which, according to that writer, was discovered by Pythagoras himself, and is esteemed by Plutarch as «unquestionably more subtle and more scientific than the theorem which demonstrated that the square on the hypotenuse is equal to the squares on the sides about the right angle»⁽¹⁾. This proposition enables us *to apply to a given straight line in a given rectilineal angle, a parallelogram equal to a given triangle*.

The solution depends on the preceding theorem (Euclid I, 43) which states that *in any parallelogram the complements of the parallelogram about the diameter are equal to one another*. In view of the interpretation I put on the present passage, and considering the fact that Pythagoras appears to have only considered squares described on the sides of a right-angled triangle, and not parallelograms or other figures, or a general triangle, as was done by some of his successors⁽²⁾, I would suggest that the early Pytha-

(1) Heath, *loc. cit.*

(2) Proclus would seem to attribute this to Euclid himself, who at any rate chronicles it in the 31 st proposition of his VI the book.

goreans confined their attention to the case where the given rectilinear angle was a right angle, and that the consideration of other angles was a later innovation, with which Plato was unfamiliar. In what follows accordingly I make the assumption that the *parallelograms are all rectangles*.

We have in Fig. 1⁽¹⁾ the graphical solution of the problem of applying to a given line $\alpha\beta$ a rectangle $\alpha\mu$, equal in area

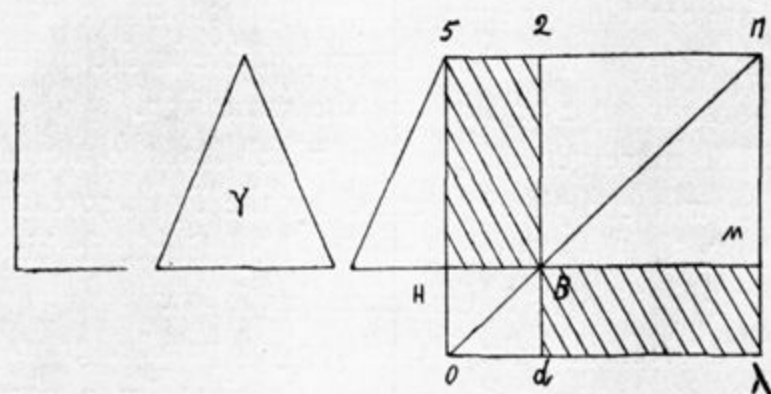


Fig. 1

to a given triangle γ . Bisecting one side of γ in the point H , we place γ with that side as the perpendicular to $\alpha\beta$ at β , and the opposite vertex pointing away from α . We then construct the rectangle with sides parallel and perpendicular to $\alpha\beta$, one side passing through the vertex just referred to and the adjacent side through H , and complete the figure. The shaded rectangle $H\epsilon$ is then so constructed as to be equal in area to the given triangle, and is also equal to the other shaded area $\alpha\mu$, since the two shaded rectangles are precisely what were called *the complements about the diagonal $\theta\pi$* of the large rectangle. The shaded rectangle $\alpha\mu$ gives therefore the graphical solution of the problem.

(1) Traced from the first printed edition (Editio Princeps, referred to below as Ed. Pr) of Euclid, (Bâle 1553) the shading being added & the figure turned through a right angle for the purposes of the sequel. It will be noticed that the given angle is here a right angle.

Algebraically we have

$$ax = b,$$

where b is the area of the given triangle and a the length of the given line.

The problem whose solution filled Plutarch with enthusiasm is accordingly equivalent to the general equation of the first degree, which, is always soluble in the manner of Pythagoras when a and b have the same sign.

5. The more general problem which leads to a quadratic equation was completely treated by the Pythagoreans, and their work is to be found chronicled by Euclid in his Sixth Book. Proposition 28 is enunciated as follows:

«To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammatic figure similar to a given one».

The graphic solution of this problem cannot always be effected, and the Pythagoreans had determined *under what hypothesis it would be possible*.

«Thus», says Euclid, «the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect».

The necessity of this hypothesis had been proved by Euclid in the preceding theorem, (VI, 27).

Fig. 2, in which AB is the given line and AA' the pattern rectangle, gives the graphical solution, supposing the hypothesis verified.

On the other hand it shews the necessity of the hypothesis, if the problem is set in a form which can be solved. But more, it shews that, if one solution is possible, there is an alternative solution given simultaneously; this is indicated in the figure by the shading of the two rectangles AQ and $A'Q$ and by the lettering of the figure.

Indeed inspection shews that each of the rectangles

$BG, BQ, B'Q$ and GQ

whose opposite corners lie on the main diagonal BB' , are similar to the pattern, and that the two shaded rectangles, being complements about the main diagonal, are of equal area, while one of these AQ is divided into two parts by the central line EG , of which one can be exactly fitted on to the rectangle ER and the other is equal to the rectangle QF , by the theorem about the complements. Thus each of the shaded areas is equal to the gnomon got by removing from the rectangle GB the similar rectangle GQ .

This enables us to solve the problem with ruler & compasses when the hypothesis is verified. For then the rectangle GQ is known in area and in form, so that we can construct it in the corner of the rectangle GB and so obtain the whole diagram, with the two shaded rectangles simultaneously, giving the alternative solutions according as we regard AB or $A'B'$ as the given line. To get the second shaded rectangle into place on AB , we only need, of course, to rotate the figure round G through two right angles, it then comes into the position Aq , where $Gq = GQ$.

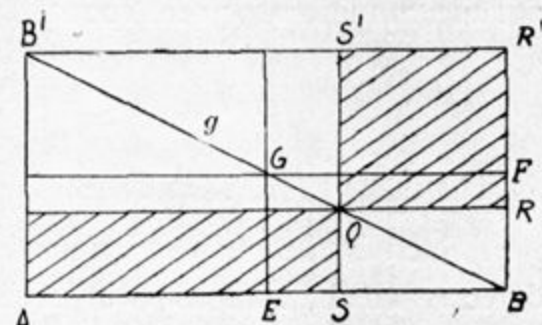


Fig. 2

On the other hand, supposing the problem solved by AQ , or by $A'Q$, we can construct this rectangle, and the vertex Q will necessarily lie on the main diagonal, by reason of the similarity of the defect to the pattern rectangle.

Thus again we get our figure, shewing that the shaded

areas, which give alternative solutions, are less than the rectangle on the half line, similar to the pattern, by the small rectangle GQ.

6. Algebraically, denoting by c the area of the given rectilinear figure, by $2a$ the length of the given line and by $2b$ the length of AB' , (the adjacent side of the pattern rectangle), finally by x_1 and x_2 the lengths AS and $A'S'$ in the figure, we have

$$x_1 + x_2 = 2a,$$

and from the equality of the shaded areas with that of the given rectilinear figure,

$$x_1 x_2 \cdot \frac{b}{a} = x_2 x_1 \cdot \frac{b}{a} = c.$$

It follows that x_1 and x_2 are the roots of the quadratic equation

$$x^2 - 2ax + ac/b = 0,$$

with the condition for real roots

$$c \leq ab,$$

which is the algebraic expression of Euclid's hypotheses.

7. Such is the geometrical problem concerned with the «defect», which, with its twin problem concerning the «excess», constituted the basis of what has been called the Geometrical Algebra, or perhaps better the Graphical Algebra, of the Greeks. We have to remember again, in connecting the account which we have given with the passage in the *Meno* before us, that we have been drawing from Euclid, who lived two hundred years after Plato, by which time the theory of Proportion had been consolidated. Tradition

connects the elaboration of a perfectly satisfactory theory of Proportion with the name of Eudoxus (390-337 B. C.) who, though much younger than Plato (429-348), was his friend and disciple⁽¹⁾ and even, perhaps, his teacher⁽²⁾ in the present connexion. The term

ὅμοιος

which is used already, but in a somewhat tentative manner, by Aristotle for what we now call *similar* in relation to geometrical figures⁽³⁾, had, by Euclid's time, become the technical term for that concept. There is, as far as I have been able to trace, no reason to suppose that Plato and his predecessors though certainly familiar with the notion of geometrical similarity, used the above word to express it technically. On the contrary, it would seem that Aristotle⁽⁴⁾ sometimes used that same word, in a kindred not, identical sense to express that two angles were *equal*, and in this sense the term seems to have been originally used by Thales.

It is not surprising therefore to find Plato in the present instance using a different term

τοιοῦτος οἶος

(1) ὁ ταῖρος δὲ τῶν περὶ Πλάτωνα, (Proclus, in his historical summary, Ed. Pr. II, p. 19).

(2) Εὐδόξου τινὸς... τοῦ Πλάτωνος διδασκάλου, (Scholion to Bk V. Ed. Pr. I, p. 54) Eva Sachs in her valuable little book «Die Fünf Platonischen Körper» (Philologische Untersuchungen herausgegeben von A. Kiessling & U. v. Wilamowitz-Moellendorf, Heft 24, Berlin, 1917) rejects this entirely as «false», (p. 89), although she notes that it stands among the Vatican Scholia, which Heath calls «the most ancient and important collection of scholia». Heath quotes the Scholion, without comment, in his introduction to Bk V.

(3) Anal. post. II, 17, 99. Quoted by Heath in a note on Euclid Bk VI.

(4) De Caelo IV., 4, 311 B 34. Quoted by Heath in a note on Euclid Bk I, Prop. 5; see also the note on Def. 10.

which in popular speech may have expressed more vividly than the later expression the exact similarity of form, without equality, or congruence, of the figures under comparison.

8. The figure (Fig. 2) used in our solution of the problem concerning the «defect» shews immediately how to solve an allied problem involving only triangles; that, in fact, in

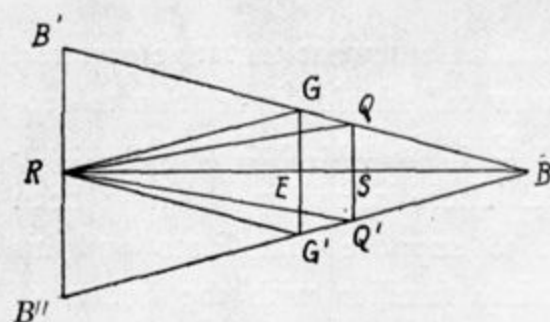


Fig. 3

which the triangle ASQ of given area $c/2$ is inset in the triangle ABB' , so that one vertex lies at A on AB' , the second at S on AB and the third at Q on BB' . Or, reflecting the figure in the line AB (Fig 3) we

have a triangle AQQ' of given area c inset in a given isosceles triangle $BB'B''$ ⁽¹⁾.

This idea of the insetting of one triangle in another is not to be found in Euclid, and it is only by considering the few words, casually dropped by Socrates in the passage before us,

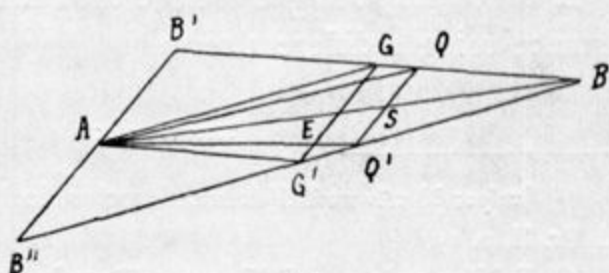


Fig. 4

that I have been led to put the problem in this way. I can imagine that the idea was Plato's own. He was fond ⁽²⁾ of varying the habitual formulation of a problem, and, by giving

⁽¹⁾ The case when the given triangle is general (Fig. 4) can of course be similarly treated, using the general form of Euclid VI, 28.

⁽²⁾ E. g. in his treatment of the Delian problem.

it a new turn of expression, he threw light, both on the question and its applications.

In the present case, with his love for circles and for symmetry, he might very well go further, and replace the triangle $BB'B''$ by a circle of diameter AB , (Fig. 5).

Was not Plato himself the $\tau\iota\varsigma$, the indefinite *someone*, who posed the problem in this form, and was not the *geometrician* his Pythagorean friend Archytas?

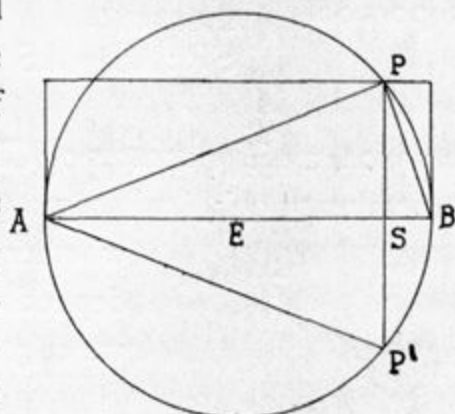


Fig. 5

«Is it possible to inset, not in a triangle but in a circle, a triangle of given area»?

9. That this problem, like the other, requires some extra hypothesis in order to render it soluble, is fairly obvious. If we suppose the process to have been performed on the same lines as before, the diameter AB of the circle being now the given line, and APP' the inset triangle with AP as equivalent rectangle, the defect BP will no longer be similar to a preassigned rectangle, independent of the given area Δ , but, by well-known properties of the circle, will vary with Δ , *being in every case similar to the rectangle AP* . This results from the fact that the angle in a semi-circle is a right angle. a fact known to Thales, two hundred years before Plato's time.

The figure before us of a right-angled triangle APB with a perpendicular PS drawn from the right angle to the hypotenuse, is one that constantly recurs in Plato's mathematical work, just as the same philosophical concepts continually recur in the different Dialogues.

He used this figure in his investigations into the Delian problem of the duplication of the cube. In the Delian problem

we require a repetition of the figure before us, obtained by drawing a perpendicular to PA from A, meeting PS produced in R, (Fig. 6), so as to enable us to visualise the continued proportion

$$SR : AS = AS : SP = SP : SB$$

on which, as Hippocrates of Chios, author of the first mathematical text-book, had shewn, the multiplication of the cube, whether by 2 or by some other number, depends. To solve the Delian problem we require to make

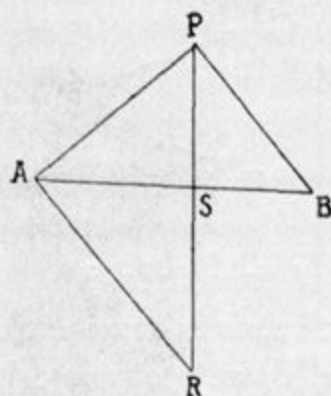


Fig. 6

$$SR = 2SB,$$

and it seems likely to me that Plato in his mathematical lectures must have used what we now call *the argument from continuity* to shew that this could be done, whether the simple mecha-

nical contrivance for effecting this, whose description has come down to us⁽¹⁾, is really to be attributed to him, or as seems more probable, to one of his pupils.

The same figure occurs, as I have shewn elsewhere⁽²⁾, in connection with the Nuptial Number of Plato.

10. The hypothesis thrown off by Socrates is seen in this way to suggest itself. We have to assume that, among the different possible ways of applying to a part of the given line AB a rectangle equal in area to that given, there is one AP such that the corresponding defect BP is similar to AP. The point P then lies of necessity on the circle on AB as diameter.

(1) Given by Eutokius of Askalon in his commentary on Archimedes.

(2) Proc. London Math. Soc. Series II, Vol. 23, (1923) pp. 27-44.

The actual identification of the hypothecated rectangle was another matter. Like the Delian problem it requires higher methods.

We may, for instance, (Fig. 7), use one invented by

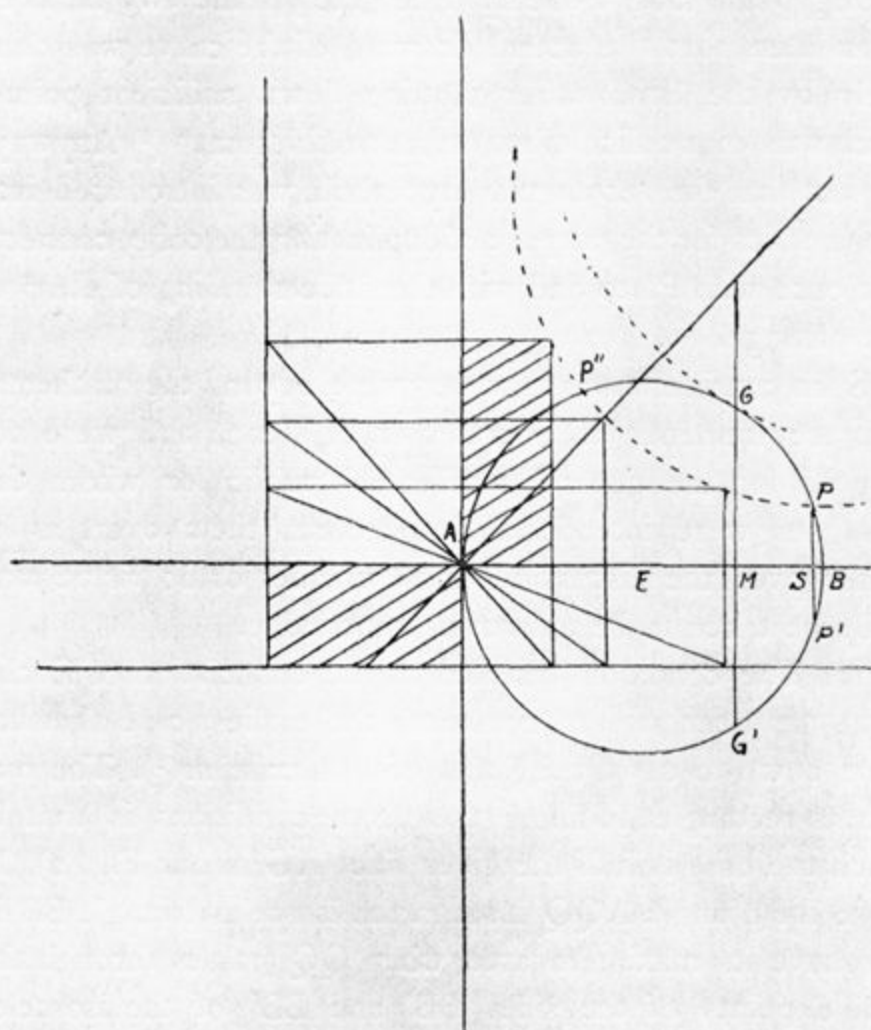


Fig. 7

Platot's pupil Menaechmus for duplicating the cube, & cut the point P out of the circle by a rectangular hyperbola with AB and the tangent at A to the circles as asymptotes. If we consider Fig. 1, in which the point β is denoted by A in Fig. 7, and if we suppose the line $\theta\pi$ to turn round the point β , each position will give us a point λ , and these trace out a

branch of a rectangular hyperbola. Menaechmus probably used some simple instrument, perhaps a pair of parallel rulers, in constructing his rectangles; but it is clear from a letter of the well-known mathematician Eratosthenes to the Egyptian king Ptolemy Euergetes, some two hundred years later, that the actual drawing of the curve was done freely by the hand, after plotting down sufficient points, which was «not difficult, but rather troublesome» ⁽¹⁾ Anyway, the fact that the rectangular hyperbola was introduced into mathematics for the purpose of duplicating the cube, connects it at once with Plato and with the understanding of the present passage.

Such considerations however were beyond the mathematical acquirements of Socrates or of Meno, and were not adapted for the general intellectual public for whom the Dialogues were intended; Plato passes them over, as he passes over the discussion as to whether or no virtue may really be taken to be a form of knowledge.

11. The condition inserted in the enunciation of Euclid VI-28 has been said to constitute the earliest case that has come down to us of the finding of a *maximum* ⁽²⁾. The maximum rectangle fulfilling the given conditions is that in which the base is half the given line (AG in our Fig. 2). In Fig. 3 the triangle AQQ' inset in the isosceles triangle BB'B'' will have its maximum area when QQ' passes through the middle points of the sides BB' and BB'' of the isosceles triangle.

In the problem which we have traced in the present investigation to Plato, the maximum triangle inset in the circle is equilateral, and corresponds in Fig. 5 to the case when PP' passes through the middle point M of the radius EB. This

⁽¹⁾ M. Cantor, *loc. cit.*, p. 212.

⁽²⁾ M. Cantor, *loc. cit.*, p. 266.

If we use for this purpose the rectangular hyperbola of Menaechmus already referred to, this curve will, when the given triangle has the maximum area, touch the circle at G, giving us an unique solution; if, however the given area is less than the maximum, the hyperbola will (Fig. 7) cut the circle in two points P and P'' one on each side of the line GG', so that, as in the Pythagorean problem the solutions appear simultaneously in pairs.

12. Our problem does not however reduce to the solution of a quadratic equation; its expression in the form of a biquadratic with at most two real roots is immediate.

Taking the diameter AB and the tangent to the circle at A as axes of x and y respectively, the equation to the circle is

$$(x-a)^2 + y^2 = a^2,$$

while, the area of the inset isosceles triangle being c , we have

$$xy = c,$$

which is the equation to the equilateral hyperbola already referred to. Eliminating y we get the required biquadratic,

$$x^4 - 2ax^3 + c^2 = Q. \quad (I)$$

If x is negative, or, being positive, is $> 2a$, the expression on the left of this equation is obviously positive, not zero; thus any real roots must lie in the interval $0 \leq x \leq 2a$. Differentiating the left-hand side, we see that it decreases monotonely from $x=0$ to $x+3a/2$ and then increases monotonely, the minimum value of the expression being $c^2 - 27a^4/16$, while at both ends of the interval $0 \leq x \leq 2a$ the expression has the value c^2 . Thus there cannot be any real root if $c^2 - 27a^4/16$ is, like c^2 , positive, that is, if

$$c > 3\sqrt{3}a^2/4,$$

(i. e. if the given area is greater than that of the equilateral triangle inscribed in the circle); if

$$c < 3\sqrt{3}a^2/4,$$

the expression on the left of (1) passes monotonely from positive to negative and then from negative to positive, and therefore passes twice through the value zero, so that there are two real roots one less than $3\sqrt{3}a^2/4$ and one greater than the same.

Only when c has the value $3\sqrt{3}a^2/4$ is there one and only one real root, namely $x = \frac{2}{3}a/2$, which is then a repeated root.

These analytical results exactly correspond, of course, to those obtained previously by geometry.

13. In conclusion it may be remarked that the hypothesis proposed by Socrates is not only sufficient, as we have seen, for his purpose, but is also necessary to ensure the solubility of the problem. If, as I surmise, Plato was familiar with *the argument from continuity*, this would not have escaped him. Indeed, if there be any triangle XYZ, of the required area, inset in the circle, the isosceles inset triangle TYZ (Fig. 9) on the same base, and on the same side of it as the centre, will of course, be of greater area, if the point X is different from T. But, by replacing Y and Z by any other pair of points of the arc subtending the angle YTZ, and placed symmetrically with respect to its middle point, we get isosceles inset triangles

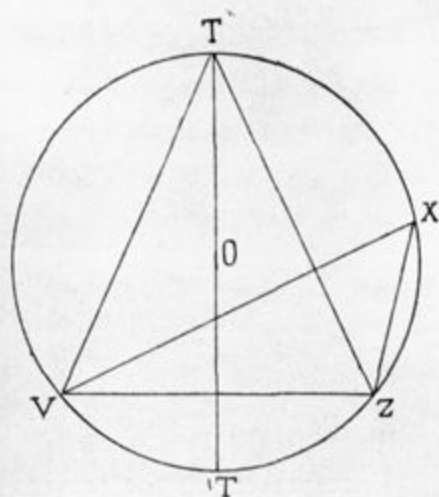


Fig. 9

of every area from zero to that of the triangle TYZ, so that one of these will have the required area.

Thus Socrates has given a condition which is sufficient and necessary, and was in strict accordance with the spirit of his times. Nowadays, unfamiliar as we are with the graphical methods of the Greeks, we might be inclined to propose as an equivalent hypothesis that the given area should be less than that of the equilateral triangle inset in the given circle.

Grace Chisholm Young.

Since presenting the present memoir in April 1927, following out the traces of the word *παρτείνεσθαι* given by Adam, *loc. cit.*, I find ample corroboration of my translation in Butcher's article (1888), *Journal of Philology*, vol. XVII, pp. 219 seq. I have only seen excerpts from this article, printed in E. S. Thompson's school edition of the *Meno*, (Macmillan and Co., London, 1901). I need hardly say that my views on the value of Plato's mathematics are not in accordance with those set forth in Thompson's notes.